

A Generalization of the Lamb–Bateman Integral Equation and Fractional Derivatives : A Comment

Kazuyuki FUJII *

*Department of Mathematical Sciences

Yokohama City University

Yokohama, 236–0027

Japan

Abstract

In this note a generalization of the Lamb–Bateman integral equation is presented and its solution is given in terms of **fractional derivatives**. This is a comment one to the paper by Babusci, Dattoli and Sacchetti (arXiv:1006.0184 [math-ph]).

In the paper [1] the authors reconsidered the integral equation presented by Lamb [2]

$$\int_0^\infty u(x-y^2)dy = f(x), \quad (1)$$

where $f(x)$ is a function given and $u(x)$ is a function to be determined. First, an interesting solution

$$u(x) = \frac{2}{\pi} \int_{-\infty}^x \frac{f'(\xi)}{\sqrt{x-\xi}} d\xi \quad (2)$$

was given by Harry Bateman. Note that this form is a bit different from the original one. However, the proof by Bateman himself has not been known. In [1] a modern derivation in

*E-mail address : fujii@yokohama-cu.ac.jp

terms of **fractional derivatives** was given to be

$$u(x) = \frac{2}{\sqrt{\pi}} \partial_x^{1/2} f(x) = \frac{2}{\pi} \int_{-\infty}^x \frac{f'(\xi)}{\sqrt{x-\xi}} d\xi, \quad (3)$$

which is instructive enough. Here we set $\partial_x = \frac{d}{dx}$ for simplicity. In the following this notation is used.

We would like to generalize the equation (1). In [1] a generalization was presented to be

$$\int_0^\infty u(x - y^m) dy = f(x) \quad (m \geq 2). \quad (4)$$

However, we present another generalization. By rewriting (1) to be

$$\int_{-\infty}^\infty u(x - y^2) dy = f(x),$$

(the coefficient 1/2 has been omitted for simplicity) we give a multi-dimensional integral equation

$$\int \int \cdots \int_{\mathbf{R}^n} u(x - \sum_{j=1}^n y_j^2) dy_1 dy_2 \cdots dy_n = f(x), \quad (5)$$

where the function f is assumed to be “good” (which means that f has all properties required in the process of calculation).

Let us solve the equation in a similar way as in [1]. The polar coordinates of \mathbf{R}^n

$$\mathbf{r} = (r, \theta_{n-2}, \cdots, \theta_2, \theta_1, \phi) \longrightarrow \mathbf{y} = (y_1, y_2, y_3, \cdots, y_{n-1}, y_n)$$

is given by ($0 \leq r < \infty$, $0 \leq \theta_{n-2}, \cdots, \theta_2, \theta_1 \leq \pi$, $0 \leq \phi < 2\pi$)

$$\begin{aligned} y_n &= r \cos \theta_{n-2} \\ y_{n-1} &= r \sin \theta_{n-2} \cos \theta_{n-3} \\ &\vdots \\ y_3 &= r \sin \theta_{n-2} \cos \theta_{n-3} \cdots \sin \theta_2 \cos \theta_1 \\ y_2 &= r \sin \theta_{n-2} \cos \theta_{n-3} \cdots \sin \theta_2 \sin \theta_1 \sin \phi \\ y_1 &= r \sin \theta_{n-2} \cos \theta_{n-3} \cdots \sin \theta_2 \sin \theta_1 \cos \phi. \end{aligned} \quad (6)$$

The Jacobian of this coordinate transformation is given by

$$\mathbf{J} = \det \left(\frac{\partial \mathbf{y}}{\partial \mathbf{r}} \right) = \pm r^{n-1} \sin^{n-2}(\theta_{n-2}) \sin^{n-3}(\theta_{n-3}) \cdots \sin \theta_1. \quad (7)$$

Note that the proof is not so easy for undergraduates.

Then, under the coordinate transformation the equation (5) becomes

$$\text{Vol}(S^{n-1}) \int_0^\infty r^{n-1} u(x - r^2) dr = f(x) \quad (8)$$

where $\text{Vol}(S^{n-1})$ is the volume of the $(n-1)$ -dimensional sphere S^{n-1} given by

$$\begin{aligned} \text{Vol}(S^{n-1}) &= \int_0^\pi \sin^{n-2}(\theta_{n-2}) d\theta_{n-2} \int_0^\pi \sin^{n-3}(\theta_{n-3}) d\theta_{n-3} \cdots \int_0^\pi \sin \theta_1 d\theta_1 \int_0^{2\pi} 1 d\phi \\ &= B\left(\frac{n-1}{2}, \frac{1}{2}\right) B\left(\frac{n-2}{2}, \frac{1}{2}\right) \cdots B\left(\frac{3}{2}, \frac{1}{2}\right) B\left(\frac{2}{2}, \frac{1}{2}\right) \times 2\pi \\ &= 2\pi \frac{\Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n-2}{2}\right) \Gamma\left(\frac{1}{2}\right) \cdots \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{2}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n-1}{2}\right) \cdots \Gamma\left(\frac{4}{2}\right) \Gamma\left(\frac{3}{2}\right)} \\ &= 2\pi \frac{\Gamma\left(\frac{1}{2}\right)^{n-2}}{\Gamma\left(\frac{n}{2}\right)} \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}, \end{aligned}$$

where the calculation is based on the following properties

$$\int_0^\pi \sin^m \theta d\theta = B\left(\frac{m+1}{2}, \frac{1}{2}\right)$$

and

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}; \quad \Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}$$

where $B(p, q)$ and $\Gamma(p)$ are respectively the Beta-function and Gamma-function given by

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx, \quad \Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx.$$

Tentatively, by setting $C = \text{Vol}(S^{n-1})$ we have

$$\int_0^\infty r^{n-1} u(x - r^2) dr = \frac{1}{C} f(x). \quad (9)$$

In terms of the formula of Taylor expansion (which is formal)

$$e^{a\partial_x} g(x) = g(x + a)$$

the left hand side of (9) becomes

$$\text{LHS} = \int_0^\infty r^{n-1} e^{-r^2 \partial_x} u(x) dr = \left\{ \int_0^\infty r^{n-1} e^{-r^2 \partial_x} dr \right\} u(x),$$

so we set $a = \partial_x$ **formally** and calculate the integral

$$\int_0^\infty r^{n-1} e^{-r^2 a} dr = \int_0^\infty r^{n-1} e^{-ar^2} dr.$$

It is easily performed by setting $t = ar^2$ ($\Rightarrow r = (t/a)^{1/2}$) and becomes

$$\begin{aligned} \int_0^\infty r^{n-1} e^{-ar^2} dr &= \int_0^\infty \left(\frac{t}{a} \right)^{\frac{n-1}{2}} e^{-t} \frac{dt}{2\sqrt{at}} \\ &= \frac{1}{2} \left(\frac{1}{a} \right)^{\frac{n}{2}} \int_0^\infty t^{\frac{n}{2}-1} e^{-t} dt \\ &= \frac{1}{2} \Gamma\left(\frac{n}{2}\right) a^{-\frac{n}{2}}. \end{aligned}$$

Here, by inserting ∂_x in place of a

$$\left\{ \frac{1}{2} \Gamma\left(\frac{n}{2}\right) \partial_x^{-\frac{n}{2}} \right\} u(x) = \frac{1}{C} f(x)$$

and we obtain the (formal) solution

$$u(x) = \frac{2}{C \Gamma\left(\frac{n}{2}\right)} \partial_x^{\frac{n}{2}} f(x). \quad (10)$$

The explicit value of C gives

$$C = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \implies \frac{2}{C \Gamma\left(\frac{n}{2}\right)} = \pi^{-\frac{n}{2}},$$

so we have a good-looking form

$$u(x) = \pi^{-\frac{n}{2}} \partial_x^{\frac{n}{2}} f(x). \quad (11)$$

Now we divide n into two cases :

(a) $n = 2m$

$$u(x) = \pi^{-m} \partial_x^m f(x) = \pi^{-m} f^{(m)}(x). \quad (12)$$

In this case there is no problem.

(b) $n = 2m + 1$ In the case

$$u(x) = \pi^{-(m+\frac{1}{2})} \partial_x^{m+\frac{1}{2}} f(x),$$

so we encounter the fractional derivatives. Since

$$\begin{aligned} u(x) &= \pi^{-(m+\frac{1}{2})} \partial_x^{m+\frac{1}{2}} f(x) \\ &= \pi^{-(m+\frac{1}{2})} \partial_x^{-\frac{1}{2}+m+1} f(x) \\ &= \pi^{-(m+\frac{1}{2})} \partial_x^{-\frac{1}{2}} \partial_x^{m+1} f(x) \\ &= \pi^{-(m+\frac{1}{2})} \partial_x^{-\frac{1}{2}} f^{(m+1)}(x), \end{aligned}$$

we can use the following well-known formula [4], [5]

$$\partial_x^{-\mu} g(x) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x g(\xi) (x - \xi)^{\mu-1} d\xi \quad (\mu > 0). \quad (13)$$

By inserting $\mu = \frac{1}{2}$ into the formula we obtain ($\Gamma(1/2) = \sqrt{\pi}$)

$$\begin{aligned} u(x) &= \pi^{-(m+\frac{1}{2})} \frac{1}{\Gamma(\frac{1}{2})} \int_{-\infty}^x f^{(m+1)}(\xi) (x - \xi)^{\frac{1}{2}-1} d\xi \\ &= \pi^{-(m+1)} \int_{-\infty}^x \frac{f^{(m+1)}(\xi)}{\sqrt{x - \xi}} d\xi. \end{aligned} \quad (14)$$

Summary

Equation :

$$\int \int \cdots \int_{\mathbf{R}^n} u(x - \sum_{j=1}^n y_j^2) dy_1 dy_2 \cdots dy_n = f(x).$$

Solution :

(a) $n = 2m$

$$u(x) = \pi^{-m} f^{(m)}(x).$$

(b) $n = 2m + 1$

$$u(x) = \pi^{-(m+1)} \int_{-\infty}^x \frac{f^{(m+1)}(\xi)}{\sqrt{x - \xi}} d\xi.$$

A comment is in order. We can moreover generalize the integral equation (5) as follows.

$$\int \int \cdots \int_{\mathbf{R}^n} u(x - \mathbf{y}^t A \mathbf{y}) dy_1 dy_2 \cdots dy_n = f(x) \quad (15)$$

where $\mathbf{y} = (y_1, y_2, \cdots, y_n)^t$ and A is positive-definite. However, calculation will be left to readers.

In the note we gave a generalization of the Lamb–Bateman integral equation and solved it in terms of fractional derivatives. We can say that theory of fractional derivatives (more generally, **Fractional Calculus**) is a powerful tool for solving integral equations.

References

- [1] D. Babusci, G. Dattoli and D. Sacchetti : The Lamb–Bateman Integral Equation and the Fractional Derivatives, arXiv:1006.0184 [math-ph].
- [2] H. Lamb : On the diffraction of a solitary wave, Proc. London Math. Soc. **8** (1910), 422.
- [3] E. T. Whittaker and G. N. Watson : A Course of Modern Analysis, Cambridge University Press (since 1902).
- [4] K. Oldham and J. Spanier : The Fractional Calculus, 2006, Dover Publications.
- [5] Akira Asada : Fractional calculus and infinite order differential operator, Yokohama Math. J. **55** (2010), 129.